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Abstract. Hybrid systems are a powerful formalism for modeling and reasoning about cyber-physical systems. They mix the continuous and discrete natures of the evolution of computerized systems. Switched systems are a special kind of hybrid systems, with restricted discrete behaviours: those systems only have finitely many different modes of (continuous) evolution, with isolated switches between modes. Such systems provide a good balance between expressiveness and controllability, and are thus in widespread use in large branches of industry such as power electronics and automotive control. The control law for a switched system defines the way of selecting the modes during the run of the system. Controllability is the problem of (automatically) synthezing a control law in order to satisfy a desired property, such as safety (maintaining the variables within a given zone) or stabilisation (confinement of the variables in a close neighborhood around an objective point). In order to compute the control of a switched system, we need to compute the solutions of the differential equations governing the modes. Euler's method is the most basic technique for approximating such solutions. We present here an estimation of the Euler's method local error, using the notion of "one-sided Lispchitz constant" for modes. This yields a general control synthesis approach which can encompass several features such as bounded disturbance and compositionality.

1 Introduction

In this paper, we present some recent results obtained for the control synthesis of nonlinear switched systems using the one-sided Lipschitz conditions of their dynamics. The main idea is to use "one-sided Lipschitz conditions" on the system vector fields in order to generate a sequence of balls enclosing the sets of trajectories. The method can be easily extended to take into account uncertainty and compositional synthesis. These results mainly originate from collaboration with A. Le Coënt, F. De Vuyst, L. Chamoin, J. Alexandre dit Sandretto and A. Chapoutot (see [11,10]).

The plan of the paper is as follows: in Section 2, we present the notions of switched systems and (R, S)-stability; in Section 3, we introduce a new error analysis for Euler's method, and explain how to use it for ensuring (R, S)-stability in control synthesis of switched systems; we extend this control synthesis method to uncertain switched systems, and to compositional synthesis (Section 4); we conclude in Section 5.

2 Switched systems and (R,S)-stability

2.1 Switched systems

A hybrid system is a system where the state evolves continuously according to several possible *modes*, and where the change of modes (switching) is done instantaneously. We consider here the special case of hybrid systems called "sampled switched systems" where the change of modes occurs periodically with a period of τ seconds. We will suppose furthermore that the state keeps its value when the mode is changed (no jump). More formally, we denote the state of the system at time t by $x(t) \in \mathbb{R}^n$. The set of modes $U = \{1, \dots, N\}$ is *finite*. With each mode $j \in U$ is associated a vector field f_j that governs the state $x(t)$; we have:

$$\dot{x}(t) = f_j(x(t))$$

We make the following hypothesis:

$(H0)$ For all $j \in U$, f_j is a locally Lipschitz continuous map.

We will denote by $\phi_j(t; x^0)$ the solution at time t of the system

$$\begin{aligned} \dot{x}(t) &= f_j(x(t)), \\ x(0) &= x^0. \end{aligned} \tag{1}$$

The existence of ϕ_j is guaranteed by assumption $(H0)$. Let us consider $S \subset \mathbb{R}^n$ be a compact and convex set, typically a "box" or "rectangular set", that is a cartesian product of n closed intervals. We know by $(H0)$ that there exists a constant $L_j > 0$ such that:

$$\|f_j(y) - f_j(x)\| \le L_j \|y - x\| \quad \forall x, y \in S. \tag{2}$$

We also define, for all $j \in U$:

$$C_j = \sup_{x \in S} L_j \|f_j(x)\|. \tag{3}$$

Example 1. One consider the example (adapted from [9]) of a two rooms apartment, with one heater per room. See Figure 1. There is heat exchange between the two rooms and with the environment. The objective is to control the temperature of the two rooms. The continuous dynamics of the system is given by the equation:

$$\dot{\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}} = \begin{pmatrix} -\alpha_{21} - \alpha_{e1} - \alpha_f j_1 & \alpha_{21} \\ \alpha_{12} & -\alpha_{12} - \alpha_{e2} - \alpha_f j_2 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} + \begin{pmatrix} \alpha_{e1} T_e + \alpha_f T_f j_1 \\ \alpha_{e2} T_e + \alpha_f T_f j_2 \end{pmatrix}.$$

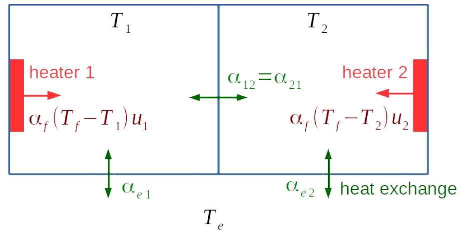

Fig. 1. 2-rooms example

Here the state of the system is (T_1, T_2) where T_1 and T_2 are the temperatures of the two rooms. The control mode of the system is of the form $j = (j_1, j_2)$ where variable j_1 (respectively j_2) can take the values 0 or 1 depending on whether the heater in room 1 (respectively room 2) is switched off or switched on (hence $U = U_1 \times U_2 = \{0,1\} \times \{0,1\}$). T_e corresponds to the temperature of the environment, and T_f to the temperature of the heaters. The values of the different parameters are the following: $\alpha_{12} = 5 \times 10^{-2}$, $\alpha_{21} = 5 \times 10^{-2}$, $\alpha_{e1} = 5 \times 10^{-3}$, $\alpha_{e2} = 5 \times 10^{-3}$, $\alpha_f = 8.3 \times 10^{-3}$, $T_e = 10$ and $T_f = 35$. We suppose that the heaters can be switched periodically at sampling instants $\tau, 2\tau, \ldots$ with $\tau = 5s$. The objective is to stabilize the state (T_1, T_2) of the system in the neighborhood of the region $R = [18, 22] \times [18, 22]$.

A *pattern* π is a finite sequence of modes; e.g., the expression $\left(\begin{pmatrix}0\\1\end{pmatrix} \cdot \begin{pmatrix}0\\0\end{pmatrix} \cdot \begin{pmatrix}1\\1\end{pmatrix}\right)$ is a pattern in Example 1. The *(state-dependent) control synthesis problem* consists in finding at each sampling time τ, 2τ, $\ldots$, the appropriate *mode* $u \in U$ (in function of the current value of x) to be selected for satisfying some objective, for example a *safety* property. More generally, the control synthesis problem (with a "time-horizon" bounded by a positive integer K) consists first in selecting at time 0 a pattern π_1 of length, say $1 \le k_1 \le K$, according to the value of state $x(0)$; then after $k_1\tau$ seconds, selecting a new pattern π_2, according to the value of $x(k_1\tau)$, and so on repeatedly. This induces a control (or switching) rule σ which is a piecewise constant function of time, with discontinuities occurring at sampling times. By convention, the control law σ is right-continuous.

2.2 (R, S)-stability

Among the classical objectives that one is generally aiming for, there are

- the *reachability* objective: given an initial region R_{init} and a *target* region R, find a pattern which drives $x(t)$ to R, for any initial state $x^0 = x(0) \in R_{init}$;
- the *stability* objective: for any initial point $x^0 = x(0) \in R$, find a pattern $\pi \in U^k$ (with $1 \le k \le K$) which makes the trajectory return in R (i.e.: $x(k\tau) \in R$) while always maintaining $x(t)$ in a *neighborhood* $S = R + \varepsilon$ of R, (i.e.: $x(t) \in S$ for $0 \le t \le k\tau$).

The effect of such control rules is depicted on Figure 2.

For the sake of simplicity, we focus here on a property that we call "(R, S)-stability": given two rectangular sets (i.e., cartesian products of intervals) R and S with $R \subseteq S \subset$

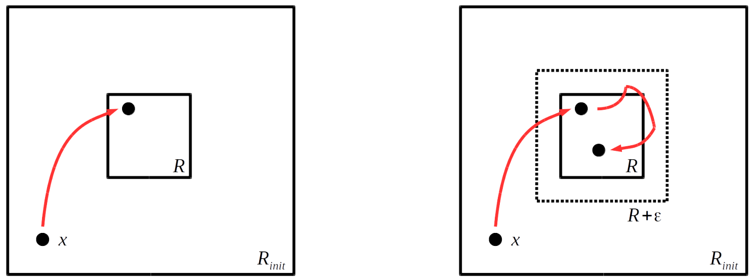

Fig. 2. Illustration of reachability (left) followed by stability (right)

$\mathbb{R}^n$, called respectively "recurrence set" and "safety set", the (R, S)-*stability control problem* consists in finding a control σ ensuring, for all $x(0) \in R$

1. *recurrence*: the state of the system $x(t)$ belongs to R for an infinite number of values of t;
2. *safety*: the state of the system $x(t)$ belongs to S for all $t \geq 0$.

The property of (R, S)-stability is illustrated in Figure 3 in the case of Example 1, with $R = [18, 22] \times [18, 22]$.

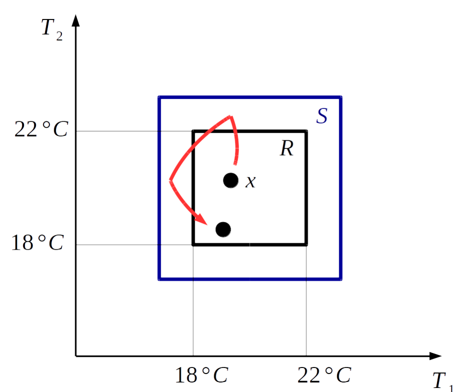

Fig. 3. (R, S)-stability

We now give the general scheme of control synthesis that has been proposed in MINIMATOR [8] for ensuring (R, S)-stability. This scheme consists in two steps:

1. *cover* R via a finite number m of subsets $B_1^0, B_2^0, ..., B_m^0$ of S (with $R \subset \bigcup_{i=1}^m B_i^0 \subseteq S$);

2. for each B_i^0 ($1 \leq i \leq m$), *find a pattern* π_i of length $k_i \leq K$ such that, starting at $t = 0$ from any point of B_i^0, the trajectory $x(t)$ controlled by π_i satisfies:

$$x(t) \in S \quad \text{for all } t \in [0, k_i\tau] \quad \wedge \quad x(t) \in R \quad \text{for } t = k_i\tau.$$

Note that, when the system returns to R (after application of some pattern) at time, say $t = t_1$, the state $x(t_1)$ belongs to $B_{i_1}^0$ for some $1 \leq i_1 \leq m$; the pattern π_{i_1} is then applied, which makes the system return to R at time $t_2 = t_1 + k_{i_1}\tau$, and so on iteratively.

Remark 1. Let us give a rough estimation of the complexity of MINIMATOR scheme. Let N be the number of modes, n the state dimension, K the time-horizon (or maximum length of patterns), $m = 2^{nd}$ the number of modes (assuming a uniform covering obtained by bisection of depth d); the MINIMATOR scheme consists essentially in enumerating all the possible patterns of length $\leq K$ until finding, for each B_i^0 ($1 \leq i \leq m$) a safe recurrent candidate; a simple calculation shows that there are $2^{nd}N^K$ candidate patterns; the complexity of the MINIMATOR scheme is thus *exponential* in n, d, K (note that the number of modes N may be itself exponential in the dimension n: for example, in a classical n-room heating example with one heater per room and two modes by heater, there are $N = 2^n$ modes).

Remark 2. Note that the set of trajectories starting at points of R form a (positive) *invariant set* included into S. There are classical methods for generating (maximal) invariant sets included into S ([3,5]). Unfortunately, these general methods are based on a *backward* reachability constructs, which, as noticed by I.M. Mitchell [13], "are more likely to suffer from numerical stability issues, especially in systems with significant contraction – the very systems where forward simulation and reachability are most effective". The forward analysis used by the MINIMATOR scheme (application of patterns) avoids such a difficulty.

2.3 Guaranteed integration

The MINIMATOR paradigm described in Section 2.2 relies implicitly on the existence of a process for overapproximating the set of trajectories originating from a subset B_i^0 during a multiple of sampling periods. Such a process is called "guaranteed integration" (or "set-integration"). As said in [15]:

"Methods of *guaranteed integration* are methods capable to compute bounds that are guaranteed to contain the solution of a given ODE at points t_j, $j = 1, 2, \ldots, m$ in the interval $(t_0, t_m]$ for some $t_m > t_0$. These methods are usually based on Taylor series or extension of Hermite-Obreschkoff schemes to interval methods. They usually consist of two phases. On an integration step from t_{j-1} to t_j, the first phase validates existence and uniqueness of the solution of (1) for all $[t_{j-1}, t_j]$ and computes a priori bounds for this solution for al $t \in [t_{j-1}, t_j]$, [19,20]; and the second phase compute tight bounds for the solution of (1) at t_j. Note that a major problem in the second phase is the *wrapping effect* [16]. It occurs when a solution set that is not a box in $\mathbb{R}^n$, $n \geq 2$, is enclosed, or wrapped, by a box on each integration step. (...) As a result of such a wrapping, an overestimation is often introduced on each integration step. Those overestimations accumulate as the integration proceeds, and the computed bounds may soon become unacceptably large. Many methods have been proposed to reduce the wrapping effect in the context of interval methods."

In order to avoid such a wrapping effect, we proposed an alternate method which, instead of using interval arithmetic [14] and higher order Taylor series, has simply recourse to the basic (forward) Euler method [11]. This is made possible through a new error analysis of the Euler method via the notion of "one-sided Lipschitz constant".

3 Euler's method and error estimation

3.1 One-sided Lipschitz constant

As remarked in [1]:

"The Lipschitz constant of [many] functions is usually region-based and often dramatically increases as the operating region is enlarged. On the other hand, even if the nonlinear system is Lipschitz in the region of interest, it is generally the case that the available observer design techniques can only stabilize the error dynamics for dynamical systems with small Lipschitz constants but fails to provide a solution when the Lipschitz constant becomes large. The problem becomes worse when dealing with stiff systems. Stiffness means that the ordinary differential equation (ODE) admits a smooth solution with moderate derivatives, together with nonsmooth ("transient") solutions rapidly converging towards the smooth ones (...) This problem has been recognized in the mathematical literature and specially in the field of numerical analysis for some time and a powerful tool has developed to overcome this problem. This tool is a generalization of the Lipschitz continuity to a less restrictive condition known as one-sided Lipschitz (OSL) continuity."

Unlike Lipschitz constants, OSL constants can be negative, which express a form of contractivity of the system dynamics. Even if the OSL constant is positive, it is in practice much lower than the Lipschitz constant [6]. The use of OSL thus allows us to obtain an upper bound for the error associated with Euler's method that is more precise than by using Lipschitz constants [11].

Let us denote by T a compact overapproximation of the image by ϕ_j of box S for $0 \leq t \leq \tau$ and $j \in U$, i.e. T is such that

$$T \supseteq \{\phi_j(t; x^0) \mid j \in U, 0 \leq t \leq \tau, x^0 \in S\}.$$

The existence of T is guaranteed by assumption $(H0)$. We now make the additional hypothesis that the vector fields f_j of the system are *one-sided Lipschitz* (OSL) [7]. Formally:

(H_U) For all $j \in U$, there exists a constant $\lambda_j \in \mathbb{R}$ such that

$$\langle f_j(y) - f_j(x), y - x \rangle \leq \lambda_j \, \|y - x\|^2 \quad \forall x, y \in T,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of two vectors of $\mathbb{R}^n$.

Remark 3. Constants λ_j as well as L_j and C_j $(j \in U)$ can be computed using constrained optimization algorithms. See Section 3.5 for details.

3.2 Euler approximate solutions

Given an initial point $\tilde{x}^0 \in S$ and a mode $j \in U$, we define the following "linear approximate solution" $\tilde{\phi}_j(t;\tilde{x}^0)$ for $t \in [0,\tau]$ by:

$$\tilde{\phi}_j(t;\tilde{x}^0) = t f_j(\tilde{x}^0) + \tilde{x}^0. \tag{4}$$

Formula (4) is nothing else but the explicit forward Euler scheme with "time step" t. It is thus a consistent approximation of order 1 in t of the exact solution of (2.1) under the hypothesis $\tilde{x}^0 = x^0$. More generally, given an initial point $\tilde{x}^0 \in S$ and pattern π of U^k, we can define a "(piecewise linear) approximate solution" $\tilde{\phi}_\pi(t;\tilde{x}^0)$ of ϕ_π at time $t \in [0; k\tau]$ as follows:

- $\tilde{\phi}_\pi(t;\tilde{x}^0) = t f_j(\tilde{x}^0) + \tilde{x}^0$ if $\pi = j \in U$, $k = 1$ and $t \in [0,\tau]$, and
- $\tilde{\phi}_\pi(k\tau + t;\tilde{x}^0) = t f_j(\tilde{z}) + \tilde{z}$ with $\tilde{z} = \tilde{\phi}_{\pi'}((k-1)\tau;\tilde{x}^0)$, if $k \geq 2$, $t \in [0,\tau]$, $\pi = j \cdot \pi'$ for some $j \in U$ and $\pi' \in U^{k-1}$.

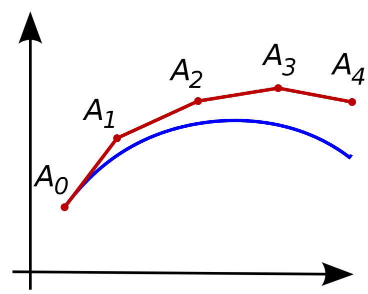

Fig. 4. Illustration of Euler's method (from Wikipedia)

We wish to synthesize a guaranteed control σ using approximate functions of the form $\tilde{\phi}_\pi$. We define the closed ball of center $x \in \mathbb{R}^n$ and radius $r > 0$, denoted $B(x,r)$, as the set $\{x' \in \mathbb{R}^n \mid \|x' - x\| \leq r\}$. Given a positive real δ^0, we now define the expression $\delta_j(t)$ which, as we will see in Theorem 1, represents (an upper bound on) the error associated to $\tilde{\phi}_j(t;\tilde{x}^0)$ (i.e. $\|\tilde{\phi}_j(t;\tilde{x}^0) - \phi_j(t;x^0)\|$).

Definition 1. *Let δ^0 be a positive constant. Let us define, for all $0 \leq t \leq \tau$, $\delta_j(t)$ as follows:*

- *if $\lambda_j < 0$:*

$$\delta_j(t) = \left((\delta^0)^2 e^{\lambda_j t} + \frac{C_j^2}{\lambda_j^2}\left(t^2 + \frac{2t}{\lambda_j} + \frac{2}{\lambda_j^2}\left(1 - e^{\lambda_j t}\right)\right)\right)^{\frac{1}{2}}$$

- *if $\lambda_j = 0$:*

$$\delta_j(t) = \left((\delta^0)^2 e^t + C_j^2(-t^2 - 2t + 2(e^t - 1))\right)^{\frac{1}{2}}$$

- *if $\lambda_j > 0$:*

$$\delta_j(t) = \left((\delta^0)^2 e^{3\lambda_j t} + \frac{C_j^2}{3\lambda_j^2}\left(-t^2 - \frac{2t}{3\lambda_j} + \frac{2}{9\lambda_j^2}\left(e^{3\lambda_j t} - 1\right)\right)\right)^{\frac{1}{2}}$$

Note that $\delta_j(t) = \delta^0$ for $t = 0$. The function $\delta_j(\cdot)$ depends implicitly on parameter: $\delta^0 \in \mathbb{R}_{>0}$. In Section 3.3, we will use the notation $\delta'_j(\cdot)$ where the value of $\delta'_j(t)$ for $t = 0$ is implicitly a parameter denoted by $(\delta')^0$.

Theorem 1. *Given an ODE system satisfying $(H0 - H_U)$, consider a point $\tilde{x}^0$ and a positive real δ^0. We have, for all $x^0 \in B(\tilde{x}^0, \delta^0)$, $t \in [0, \tau]$:*

$$\phi_j(t; x^0) \in B(\tilde{\phi}_j(t, \tilde{x}^0), \delta_j(t)).$$

The proof of this theorem is given in [11].

Remark 4. In Theorem 1, we have supposed that the step size h used in Euler's method was equal to the sampling period τ of the switching system. Actually, in order to have better approximations, it is often convenient to take a fraction of τ as for h (e.g., $h = \frac{\tau}{10}$). Such a splitting is called "sub-sampling" in numerical methods.

Corollary 1. *(*one-step invariance*) Given an ODE system satisfying $(H0 - H_U)$, consider a point $\tilde{x}^0 \in S$ and a real $\delta^0 > 0$ such that:*

1. *$B(\tilde{x}^0, \delta^0) \subseteq S$,*
2. *$B(\tilde{\phi}_j(\tau; \tilde{x}^0), \delta_j(\tau)) \subseteq S$, and*
3. *$\frac{d^2(\delta_j(t))}{dt^2} > 0$ for all $t \in [0, \tau]$.*

Then we have, for all $x^0 \in B(\tilde{x}^0, \delta^0)$ and $t \in [0, \tau]$: $\phi_j(t; x^0) \in S$.

Corollary 1 is illustrated in Figure 5. Note that condition 3 of Corollary 1 on the

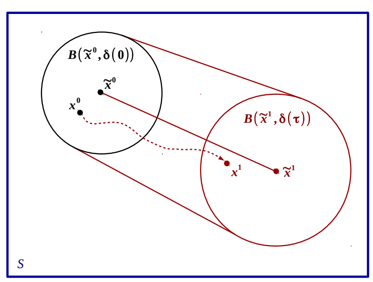

Fig. 5. Illustration of one-step invariance in S

convexity of $\delta_j(\cdot)$ on $[0, \tau]$ can be established again using an optimization function.

3.3 Application to control synthesis for (R,S)-stability

Consider a point $\tilde{x}^0 \in S$, a positive real δ^0 and a pattern π of length k. Let $\pi(k')$ denote the k'-th element (mode) of π for $1 \le k' \le k$. Let us abbreviate the k'-th approximate point $\tilde{\phi}_\pi(k'\tau;\tilde{x}^0)$ as $\tilde{x}_\pi^{k'}$ for $k'=1,...,k$, and let $\tilde{x}_\pi^{k'} = \tilde{x}^0$ for $k'=0$. It is easy to show that $\tilde{x}_\pi^{k'}$ can be defined recursively for $k'=1,...,k$, by: $\tilde{x}_\pi^{k'} = \tilde{x}_\pi^{k'-1} + \tau f_j(\tilde{x}_\pi^{k'-1})$ with $j=\pi(k')$.

Let us now define the expression $\delta_\pi^{k'}$ as follows: for $k'=0$: $\delta_\pi^{k'} = \delta^0$, and for $1 \le k' \le k$: $\delta_\pi^{k'} = \delta_j'(\tau)$ where $(\delta')^0$ denotes $\delta_\pi^{k'-1}$, and j denotes $\pi(k')$. Likewise, the expression $\delta_\pi(t)$ is defined, for $0 \le t \le k\tau$, by:

- for $t=0$: $\delta_\pi(t) = \delta^0$,
- for $0 < t \le k\tau$: $\delta_\pi(t) = \delta_j'(t')$ with $(\delta')^0 = \delta_\pi^{\ell-1}$, $j=\pi(\ell)$, $t' = t-(\ell-1)\tau$ and $\ell = \lceil \frac{t}{\tau} \rceil$.

Note that, for $0 \le k' \le k$, we have: $\delta_\pi(k'\tau) = \delta_\pi^{k'}$. Following the MINIMATOR paradigm (see Section 2.2), we are now ready to synthesize a control σ ensuring (R,S)-stability, using the approximate functions $\tilde{\phi}_\pi$.

Theorem 2. *Given a sampled switched system satisfying $(H0{-}H_U)$, consider a point $\tilde{x}^0 \in S$, a positive real δ^0 and a pattern π of length k such that, for all $1 \le k' \le k$:*

1. *$B(\tilde{x}_\pi^{k'}, \delta_\pi^{k'}) \subseteq S$ and*
2. *$\frac{d^2(\delta_j'(t))}{dt^2} > 0$ for all $t \in [0,\tau]$, with $j=\pi(k')$ and $(\delta')^0 = \delta_\pi^{k'-1}$.*

Then we have, for all $x^0 \in B(\tilde{x}^0,\delta^0)$ and $t \in [0,k\tau]$: $\phi_\pi(t;x^0) \in S$.

Corollary 2. *Given a switched system satisfying $(H0-H_U)$, consider a positive real δ^0 and a finite set of points $\tilde{x}_1, \dots \tilde{x}_m$ of S such that all the balls $B(\tilde{x}_i,\delta^0)$ cover R and are included into S (i.e. $R \subseteq \bigcup_{i=1}^m B(\tilde{x}_i,\delta^0) \subseteq S$). Suppose furthermore that, for all $1 \le i \le m$, there exists a pattern π_i of length k_i such that:*

1. *$B((\tilde{x}_i)_{\pi_i}^{k'}, \delta_{\pi_i}^{k'}) \subseteq S$, for all $k'=1,\dots,k_i-1$*
2. *$B((\tilde{x}_i)_{\pi_i}^{k_i}, \delta_{\pi_i}^{k_i}) \subseteq R$.*
3. *$\frac{d^2(\delta_j'(t))}{dt^2} > 0$ with $j=\pi_i(k')$ and $(\delta')^0 = \delta_{\pi_i}^{k'-1}$, for all $k' \in \{1,...,k_i\}$ and $t \in [0,\tau]$.*

These properties induce a control σ[1] which guarantees

- *(safety): if $x^0 \in R$, then $\phi_\sigma(t;x^0) \in S$ for all $t \ge 0$, and*
- *(recurrence): if $x^0 \in R$ then $\phi_\sigma(k\tau;x^0) \in R$ for some $k \in \{k_1,\dots,k_m\}$.*

A covering of R with balls as stated in Corollary 2 is illustrated in Figure 6 (left) with a pattern satisfying safety and recurrence in Figure 6 (right). Corollary 2 thus leads to the following method (inspired by the the MINIMATOR scheme described in Section 2.2), aiming for (R,S)-stability:

- we (pre-)compute λ_j, L_j, C_j for all $j \in U$;
- we find m points $\tilde{x}_1,\dots,\tilde{x}_m$ of S and $\delta^0 > 0$ such that $\mathbb{R} \subseteq \bigcup_{i=1}^m B(\tilde{x}_i,\delta^0)) \subseteq S$;
- we find m patterns π_i $(i=1,\dots,m)$ such that conditions 1-2-3 of Corollary 2 are satisfied.

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[1] Given an initial point $x \in R$, the induced control σ corresponds to a sequence of patterns $\pi_{i_1}, \pi_{i_2}, \dots$ defined as follows: Since $x \in R$, there exists a a point $\tilde{x}_{i_1}$ with $1 \le i_1 \le m$ such that $x \in B(\tilde{x}_{i_1},\delta^0)$; then using pattern π_{i_1}, one has: $\phi_{\pi_{i_1}}(k_{i_1}\tau;x) \in R$. Let $x' = \phi_{\pi_{i_1}}(k_{i_1}\tau;x)$; there exists a point $\tilde{x}_{i_2}$ with $1 \le i_2 \le m$ such that $x' \in B(\tilde{x}_{i_2},\delta^0)$, etc.

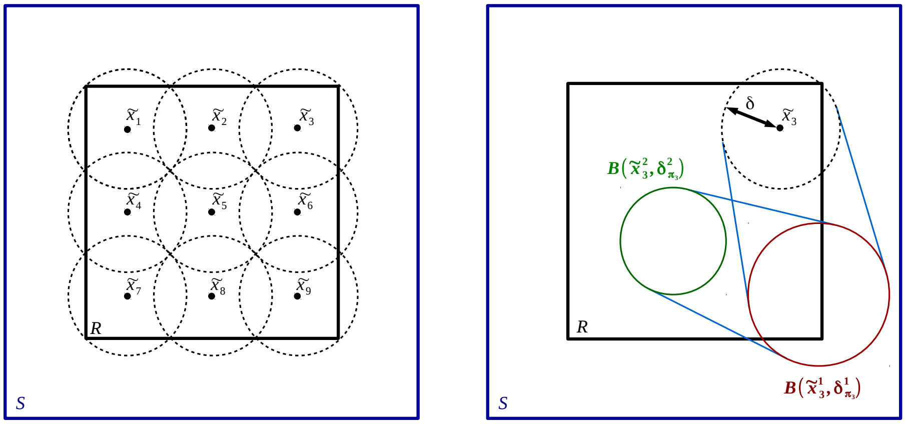

Fig. 6. Set of balls covering R (left) and safe recurrent pattern associated with one of these balls (right).

3.4 Avoiding wrapping effect with Euler's method

The problem of "wrapping effect" inherent to the method of interval analysis has been noticed from the outset: R. Moore [14] illustrates it on the simple rotation

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x; \quad x_0 \in A$$

for an initial set A which is rectangular. At each step, the rectangle is rotated and has to be wrapped by another one. At $t = 2\pi$, the blow up factor is by a factor $e^{2\pi} \approx 535$, as the step size tends to zero (Figure 7: left). In contrast, the application of the Euler-based method starting from a ball of radius 0.1 with step size 0.005, does not blow up on this example (Figure 7: right).

3.5 Numerical results

Our Euler-based synthesis method has been implemented by Adrien Le Coënt in the interpreted language Octave, and the experiments performed on a 2.80 GHz Intel Core i7-4810MQ CPU with 8 GB of memory. The computation of constants L_j, C_j, λ_j ($j \in U$) are realized with a constrained optimization algorithm. They are performed using the "sqp" function of Octave, applied on the following optimization problems:

- Constant L_j:
$$L_j = \max_{x,y \in S,\ x \neq y} \frac{\| f_j(y) - f_j(x) \|}{\| y - x \|}$$
- Constant C_j:
$$C_j = \max_{x \in S} L_j \| f_j(x) \|$$
- Constant λ_j:
$$\lambda_j = \max_{x,y \in T,\ x \neq y} \frac{\langle f_j(y) - f_j(x), y - x \rangle}{\| y - x \|^2}$$

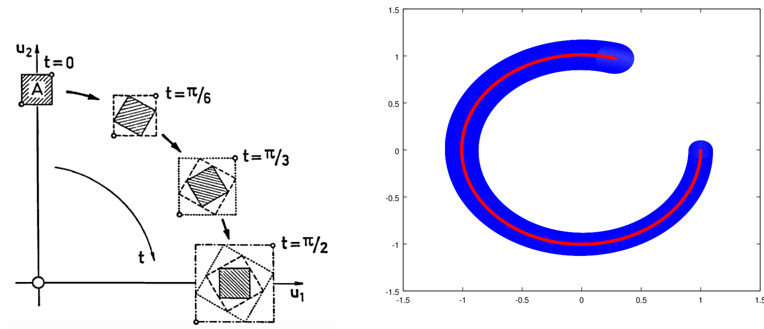

Fig. 7. left: guaranteed integration with interval method (from [14]); right: with Euler-based method.

The convexity test $\frac{d^2(\delta_j'(t))}{dt^2} > 0$ can be performed similarly. Note that in some cases, it is advantageous to use a time sub-sampling to compute the image of a ball. Indeed, because of the exponential growth of the radius $\delta_j(t)$ within time, computing a sequence of balls can lead to smaller ball images. It is particularly advantageous when a constant λ_j is negative. We illustrate this with the example of the DC-DC converter [4]. It has two switched modes, for which we have $\lambda_1 \approx -0.014$ and $\lambda_2 \approx 0.14$. In the case $\lambda_j < 0$, the associated formula $\delta_j(t)$ has the behavior of Figure 8 (a). In the case $\lambda_j > 0$, the associated formula $\delta_j(t)$ has the behavior of Figure 8 (b). In the case $\lambda_j < 0$, if the time sub-sampling is small enough, one can compute a sequence of balls with reducing radius, which makes the synthesis easier.

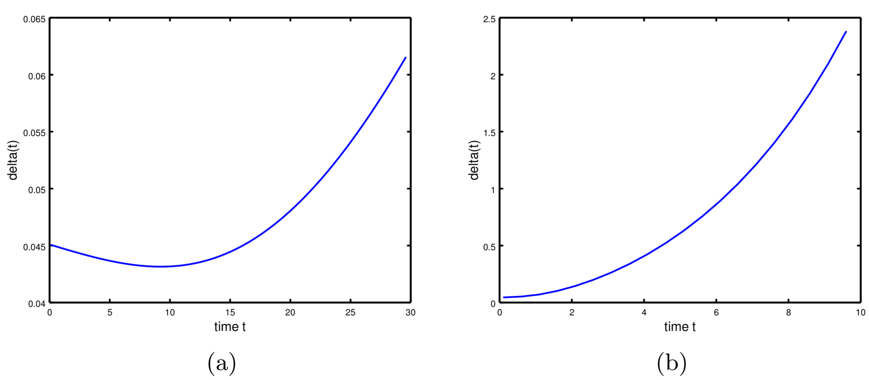

Fig. 8. Behavior of $\delta_j(t)$ for the DC-DC converter with $\delta_j(0) = 0.045$. (a) Evolution of $\delta_1(t)$ (with $\lambda_1 < 0$); (b) Evolution of $\delta_2(t)$ (with $\lambda_2 > 0$).

Example 2. (Four-room apartment) We describe a first application on a 4-room 16-switch building ventilation case study adapted from [12]. The model has been simplified in order to get constant parameters. The system is a four room apartment subject to heat transfer between the rooms, with the external environment, the underfloor, and human beings. The dynamics of the system is given by the following equation:

$$\frac{dT_i}{dt} = \sum_{j \in \mathcal{N}^* \setminus \{i\}} a_{ij}(T_j - T_i) + \delta_{s_i} b_i (T_{s_i}^4 - T_i^4) + c_i \max\left(0, \frac{V_i - V_i^*}{\bar{V}_i - V_i^*}\right)(T_u - T_i), \quad \text{for } i = 1, ..., 4.$$

The state of the system is given by the temperatures in the rooms T_i, for $i \in \mathcal{N} = \{1, \ldots, 4\}$. Room i is subject to heat exchange with different entities stated by the indices $\mathcal{N}^* = \{1, 2, 3, 4, u, o, c\}$. We have $T_0 = 30, T_c = 30, T_u = 17$, $\delta_{s_i} = 1$ for $i \in \mathcal{N}$. The (constant) parameters T_{s_i}, V_i^*, $\bar{V}_i$, a_{ij}, b_i, c_i are given in [12]. The control input is V_i ($i \in \mathcal{N}$). In the experiment, V_1 and V_4 can take the values 0V or 3.5V, and V_2 and V_3 can take the values 0V or 3V. This leads to 16 switching modes corresponding to the different possible combinations of voltages V_i. The sampling period is $\tau = 30$s. Compared simulations are given in Figure 9. On this example, the Euler-based method works better than *DynIBEX* in terms of CPU time (see Table 1).

	Euler	DynIBEX
R	$[20, 22]^2 \times [22, 24]^2$	
S	$[19, 23]^2 \times [21, 25]^2$	
τ	30	
Time subsampling	No	
Complete control	Yes	Yes
$\max_{j=1,\ldots,16} \lambda_j$	-6.30×10^{-3}	
$\max_{j=1,\ldots,16} C_j$	4.18×10^{-6}	
Number of balls/tiles	4096	252
Pattern length	1	1
CPU time	63 seconds	249 seconds

Table 1. Numerical results for the four-room example.

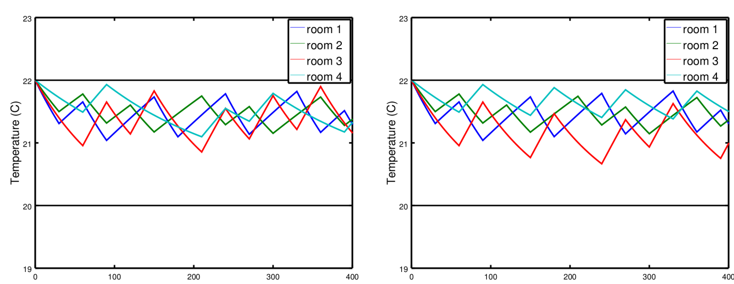

Fig. 9. Simulation of the four-room case study with Euler-based synthesis method (left) and with the synthesis method of [2] (right).

4 ODEs with uncertainty

4.1 Bounded uncertainty

Let us now consider the case where the mode j is governed by the *uncertain* ODE:

$$\dot{x}(t) = f_j(x(t), w(t)) \quad \text{with} \quad w(t) \in W$$

where W is a bounded set of diameter [2] denoted by $|W|$.

see: [Girard: Reachability of uncertain linear systems using zonotopes] [R. Alur, T. Dang, F. Ivancic, Reachability analysis of hybrid systems via predicate abstraction, Hybrid Systems : Computation and Control, C.J. Tomlin, M.R. Greenstreet (Eds), no . 2289 in LNCS, pp 35-48, 2002.] [E. Asarin, T. Dang, A. Girard, Reachability of non-linear systems using conservative approximations, Hybrid Systems : Computation and Control, O. Maler, A. Pnueli (Eds), no. 2623 in LNCS, pp 22-35, Spinger, 2003]

Let us suppose that the uncertain ODE satisfies the assumption:

$(H_{U,W})$ For all $j \in U$, there exist $\lambda_j \in \mathbb{R}$ and $\gamma_j \in \mathbb{R}_{\geq 0}$ such that, for all $x, x' \in T$, and all $w, w' \in W$:

$$\langle f_j(x,w) - f_j(x',w'), x - x' \rangle \leq \lambda_j \|x - x'\|^2 + \gamma_j \|x - x'\| \|w - w'\|.$$

Definition 2. *Let δ^0 be a positive real, and W a rectangular set of diameter $|W|$. We define, for all $j \in U$ and $0 \leq t \leq \tau$, the expression $\delta_{j,W}(t)$ as follows:*

- *if $\lambda_j < 0$,*

$$\delta_{j,W}(t) = \Bigg(\frac{C_j^2}{-\lambda_j^4} \left(-\lambda_j^2 t^2 - 2\lambda_j t + 2e^{\lambda_j t} - 2\right) + \frac{1}{\lambda_j^2} \Bigg(\frac{C_j \gamma_j |W|}{-\lambda_j} \left(-\lambda_j t + e^{\lambda_j t} - 1\right) + \lambda_j \left(\frac{\gamma_j^2 (|W|/2)^2}{-\lambda_j} (e^{\lambda_j t} - 1) + \lambda_j (\delta^0)^2 e^{\lambda_j t} \right)\Bigg)\Bigg)^{1/2} \quad (5)$$

- *if $\lambda_j = 0$,*

$$\delta_{j,W}(t) = \left(C_j^2 \left(-t^2 - 2t + 2e^t - 2\right) + \left(C_j \gamma_j |W| \left(-t + e^t - 1\right) + (\gamma_j^2 (|W|/2)^2 (e^t - 1) + (\delta^0)^2 e^t)\right)\right)^{1/2} \quad (6)$$

- *if $\lambda_j > 0$,*

$$\delta_{j,W}(t) = \frac{1}{(3\lambda_j)^{3/2}} \Bigg(\frac{C_j^2}{\lambda_j} \left(-9\lambda_j^2 t^2 - 6\lambda_j t + 2e^{3\lambda_j t} - 2\right) + 3\lambda_j \Bigg(\frac{C_j \gamma_j |W|}{\lambda_j} \left(-3\lambda_j t + e^{3\lambda_j t} - 1\right) + 3\lambda_j \left(\frac{\gamma_j^2 (|W|/2)^2}{\lambda_j} (e^{3\lambda_j t} - 1) + 3\lambda_j (\delta^0)^2 e^{3\lambda_j t} \right)\Bigg)\Bigg)^{1/2} \quad (7)$$

[2] The diameter of a set is the maximal distance of two elements.

Under assumption $(H_{U,W})$ instead of (H_U), one can naturally extend Theorem 1 and Corollary 1 to take the uncertainty set W into account, using $\delta_{j,W}(\cdot)$ in place of $\delta_j(\cdot)$. These extended results are useful to control systems with uncertainty, for example when the coefficients in the vector field definitions are known with a limited precision. Such extended forms of Theorem 1 and Corollary 1 can also be applied to control *interconnected subsystems*, each component regarding the *input* from the other one as a form of bounded uncertainty (see Section 4.2).

4.2 Application to distributed control synthesis

We now consider the distributed (or "compositional") approach which consists in splitting the original system into two sub-systems, in order to synthesize a controller σ_i ($i = 1, 2$) for each sub-system independently, then apply the control $\sigma = (\sigma_1|\sigma_2)$ (by concurrent application of σ_1 and σ_2) to the global system. The interest of the approach is to break the exponential complexity of the original method w.r.t. the dimension of the system and the number of modes (see Section 2.2). We consider an ODE of the form $\dot{x} = f_j(x)$ with $x \in \mathbb{R}^n$, $j \in U$, which is of the form

$$\dot{x}_1 = f^1_{j_1}(x_1, x_2) \tag{8}$$

$$\dot{x}_2 = f^2_{j_2}(x_1, x_2) \tag{9}$$

where the state x is of the form (x_1, x_2) with $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, the mode j is of the form (j_1, j_2), with $j_1 \in U_1$, $j_2 \in U_2$, $U = U_1 \times U_2$. Given an initial condition of the form $\begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$, and a mode $j = (j_1, j_2) \in U = U_1 \times U_2$, the solution of the ODE is now denoted by $\phi_{(j_1,j_2)}(t; x^0)$, for all $t \in [0, \tau]$. The system (8-9) can be seen as the *interconnection* of a 1st sub-system (8) where x_2 plays the role of an "input" given by (9), with a 2nd sub-system (9) where x_1 is an "input" given by (8).

Accordingly, the sets R, S and T are seen under their compositional form $R = R_1 \times R_2$, $S = S_1 \times S_2$, $T = T_1 \times T_2$. We will denote by x_1^m (resp. x_2^m) an arbitrary point of R_1 (resp. R_2), typically its central point. We denote by $L^1_{j_1}$ the Lipschitz constant for sub-system 1 under mode j_1:

$$\|f^1_{j_1}(x_1, x_2) - f^1_{j_1}(y_1, y_2)\| \le L^1_{j_1} \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|$$

We introduce also the constant:

$$C^1_{j_1} = \sup_{x_1 \in S_1} L^1_{j_1} \|f^1_{j_1}(x_1, x_2^m)\|$$

Similarly, we define the constants for sub-system 2:

$$\|f^2_{j_2}(x_1, x_2) - f^2_{j_2}(y_1, y_2)\| \le L^2_{j_2} \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|$$

and

$$C^2_{j_2} = \sup_{x_2 \in S_2} L^2_{j_2} \|f^2_{j_2}(x_1^m, x_2)\|$$

In the following, we assume that, for all $j_1 \in U_1$, there exist a real λ_{j_1} and a non-negative real γ_{j_1} which make the 1st sub-system satisfy assumption (H_{U_1,W_2}) for some overapproximation W_2 of T_2. Symmetrically, we assume that, for all $j_2 \in U_2$, there exist a real λ_{j_2} and a non-negative real γ_{j_2} which make the 2nd sub-system satisfy (H_{U_2,W_1}) for some overapproximation W_1 of T_1.

Given two modes $j_1 \in U_1, j_2 \in U_2$, and two initial conditions $\tilde{x}_1^0, \tilde{x}_2^0$, we define the "decompositional" Euler approximate solutions $\tilde{\phi}_{j_1}^1$ and $\tilde{\phi}_{j_2}^2$, for $t \in [0, \tau]$, as follows:

$$\tilde{\phi}_{j_1}^1(t; \tilde{x}_1^0) = \tilde{x}_1^0 + t f_{j_1}^1(\tilde{x}_1^0, x_2^m) \tag{10}$$

$$\tilde{\phi}_{j_2}^2(t; \tilde{x}_2^0) = \tilde{x}_2^0 + t f_{j_2}^2(x_1^m, \tilde{x}_2^0) \tag{11}$$

We can now give the distributed version of Theorem 1.

Theorem 3. *Given a distributed sampled switched system satisfying, suppose that the 1st and 2nd sub-systems satisfy, for all $j_1 \in U_1$ and $j_2 \in U_2$, the assumptions (H_{U_1,W_2}) and (H_{U_2,W_1}) respectively. Consider a point $\tilde{x}_1^0$ and a positive real δ^0. We have, for all $x_1^0 \in B(\tilde{x}_1^0, \delta^0)$, $t \in [0, \tau]$, $j_1 \in U_1$:*

$$\phi_{(j_1,j_2)}(t; x^0)_{|1} \in B(\tilde{\phi}_{j_1}^1(t, \tilde{x}_1^0), \delta_{j_1,W_2}(t)) \quad \forall j_2 \in U_2, \forall x_2^0 \in S_2, x^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}.$$

Likewise, we have, for all $x_2^0 \in B(\tilde{x}_2^0, \delta^0)$, $t \in [0, \tau]$, $j_2 \in U_2$:

$$\phi_{(j_1,j_2)}(t; x^0)_{|2} \in B(\tilde{\phi}_{j_2}^2(t, \tilde{x}_2^0), \delta_{j_2,W_1}(t)) \quad \forall j_1 \in U_1, \forall x_1^0 \in S_1, x^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}.$$

The proof of this theorem is in [10]. We can now state the distributed version of Corollary 2.

Corollary 3. *Given a positive real δ^0, consider two sets of points $\tilde{x}_1^1, \ldots, \tilde{x}_{m_1}^1$ and $\tilde{x}_1^2, \ldots, \tilde{x}_{m_2}^2$ such that all the balls $B(\tilde{x}_{i_1}^1, \delta^0)$ and $B(\tilde{x}_{i_2}^2, \delta^0)$, for $1 \leq i_1 \leq m_1$ and $1 \leq i_2 \leq m_2$, cover R_1 and R_2. Suppose that there exist patterns $\pi_{i_1}^1$ of length k_{i_1} for the 1st sub-system such that :*

1. *$B((\tilde{x}_{i_1}^1)_{\pi_{i_1}^1}^{k'}, \delta_{\pi_{i_1}^1}^{k'}) \subseteq S_1$, for all $k' = 1, \ldots, k_{i_1} - 1$;*
2. *$B((\tilde{x}_{i_1}^1)_{\pi_{i_1}^1}^{k_{i_1}}, \delta_{\pi_{i_1}^1}^{k_{i_1}}) \subseteq R_1$;*
3. *$\frac{d^2(\delta'_{j_1}(t))}{dt^2} > 0$ with $j_1 = \pi_{i_1}^1(k')$ and $(\delta')^0 = \delta_{\pi_{i_1}^1}^{k'-1}$, for all $k' \in \{1, ..., k_{i_1}\}$ and $t \in [0, \tau]$.*

and symmetrically for the 2nd sub-system. These properties induce a control σ_1 for the 1st sub-system, and σ_2 for the 2nd sub-system such that the composed control $\sigma = (\sigma_1 | \sigma_2)$ ensures recurrence in R and safety in S, i.e.:

- *if $x^0 \in R$, then $\phi_\sigma(t; x^0) \in S$ for all $t \geq 0$;*
- *if $x^0 \in R$, then $\phi_\sigma(k_{i_1}\tau; x^0)_{|1} \in R_1$ for some $i_1 \in \{1, \ldots, m_1\}$, and symmetrically $\phi_\sigma(k_{i_2}\tau; x^0)_{|2} \in R_2$ for some $i_2 \in \{1, \ldots, m_2\}$.*

Example 3. We demonstrate the interest of the distributed approach by comparing it with respect to the (centralized) approach performed in Example 2. The main difficulty of this example is the large number of modes in the switching system, which induces a combinatorial issue. The centralized controller in Example 2 was obtained with 256 balls in 48 seconds, the distributed controller was obtained with 16+16 balls in less than a second. In both cases, patterns of length 2 are used. A sub-sampling of $h = \tau/20$ is required to obtain a controller with the centralized approach (see Table 2). For the distributed approach, no sub-sampling is required for the first sub-system, while the second one requires a sub-sampling of $h = \tau/10$ (see Table 3). Simulations of the centralized and distributed controllers are given in Figure 10, where the control objective is to stabilize the temperature in $[20, 22]^4$ while never going out of $[19, 23]^4$.

	Centralized
R	$[20, 22]^4$
S	$[19, 23]^4$
τ	30
Time subsampling	$\tau/20$
Complete control	Yes
Error parameters	$\max_{j=1,\dots,16} \lambda_j = -6.30 \times 10^{-3}$ $\max_{j=1,\dots,16} C_j = 4.18 \times 10^{-6}$
Number of balls/tiles	256
Pattern length	2
CPU time	48 seconds

Table 2. Numerical results for centralized four-room example.

	Sub-system 1	Sub-system 2
R	$[20, 22]^2 \times [20, 22]^2$	
S	$[19, 23]^2 \times [19, 23]^2$	
τ	30	
Time subsampling	No	$\tau/10$
Complete control	Yes	Yes
Error parameters	$\max_{j_1=1,\dots,4} \lambda^1_{j_1} = -1.39 \times 10^{-3}$ $\max_{j_1=1,\dots,4} \gamma^1_{j_1} = 1.79 \times 10^{-4}$ $\max_{j_1=1,\dots,4} C^1_{j_1} = 4.15 \times 10^{-4}$	$\max_{j_2=1,\dots,4} \lambda^2_{j_2} = -1.42 \times 10^{-3}$ $\max_{j_2=1,\dots,4} \gamma^2_{j_2} = 2.47 \times 10^{-4}$ $\max_{j_2=1,\dots,4} C^2_{j_2} = 5.75 \times 10^{-4}$
Number of balls/tiles	16	16
Pattern length	2	2
CPU time	< 1 second	< 1 second

Table 3. Numerical results for the distributed four-room example.

5 Final remarks

We have presented a simple method of control synthesis for switched systems using a new scheme of guaranteed integration based on Euler's method. Preliminary experiments show that, on some examples, the method avoids the wrapping effect occurring

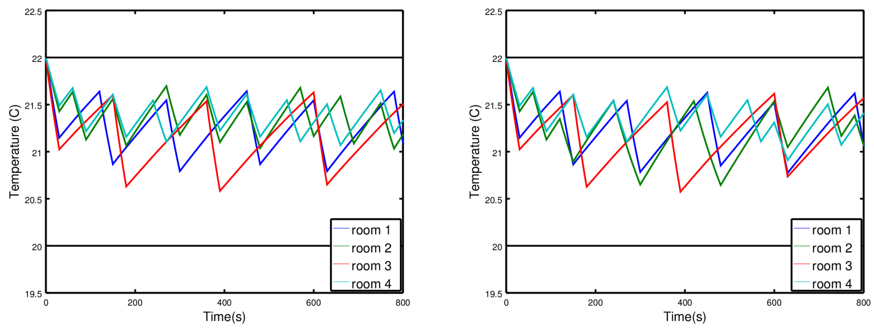

Fig. 10. Simulation of the centralized (left) and distributed (right) Euler-based controllers from the initial condition $(22, 22, 22, 22)$.

with interval-based integration methods. On-going work is done for adapting this Euler-based method to the treatment of stochastic differential equations.

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References

1. Masoud Abbaszadeh and Horacio J. Marquez. Nonlinear observer design for one-sided lipschitz systems. In *IEEE Conference on Decision and Control (CDC)*, 2010.
2. Julien Alexandre dit Sandretto and Alexandre Chapoutot. Validated Simulation of Differential Algebraic Equations with Runge-Kutta Methods. *Reliable Computing*, 22, 2016.
3. Eugene Asarin, Olivier Bournez, Thao Dang, Oded Maler, and Amir Pnueli. Effective synthesis of switching controllers for linear systems. *Proceedings of the IEEE*, 88(7):1011–1025, 2000.
4. A. Giovanni Beccuti, Georgios Papafotiou, and Manfred Morari. Optimal control of the boost dc-dc converter. In *44th IEEE Conference on Decision and Control (CDC)*, 2005.
5. F. Blanchini. Set invariance in control: a survey. *Automatica*, 35(11):1747–1768, 1999.
6. Germund Dahlquist. *Error analysis for a class of methods for stiff non-linear initial value problems*, pages 60–72. Springer Berlin Heidelberg, Berlin, Heidelberg, 1976.
7. Tzanko Donchev and Elza Farkhi. Stability and Euler approximation of one-sided lipschitz differential inclusions. *SIAM journal on control and optimization*, 36(2):780–796, 1998.
8. Laurent Fribourg, Ulrich Kühne, and Romain Soulat. Finite controlled invariants for sampled switched systems. *Formal Methods in System Design*, 45(3):303–329, 2014.

9. Antoine Girard. Low-complexity switching controllers for safety using symbolic models. In *Proceedings of 4th IFAC Conference on Analysis and Design of Hybrid Systems*, pages 82–87, 2012.
10. Adrien Le Coënt, Julien Alexandre dit Sandretto, Alexandre Chapoutot, Laurent Fribourg, Florian De Vuyst, and Ludovic Chamoin. Distributed control synthesis using Euler's method. In *International Workshop on Reachability Problems (submitted)*, 2017.
11. Adrien Le Coënt, Florian De Vuyst, Ludovic Chamoin, and Laurent Fribourg. Control synthesis of nonlinear sampled switched systems using Euler's method. In Proceedings 3rd International Workshop on *Symbolic and Numerical Methods for Reachability Analysis*, volume 247 of *Electronic Proceedings in Theoretical Computer Science*, pages 18–33. Open Publishing Association, 2017.
12. Pierre-Jean Meyer. *Invariance and symbolic control of cooperative systems for temperature regulation in intelligent buildings*. Thesis, Université Grenoble Alpes, September 2015.
13. Ian M. Mitchell. Comparing forward and backward reachability as tools for safety analysis. In *Hybrid Systems: Computation and Control, 10th International Workshop, HSCC 2007, Proceedings*, pages 428–443, 2007.
14. Ramon Moore. *Interval Analysis*. Prentice Hall, 1966.
15. Nedialko S. Nedialkov and Kenneth R. Jackson. *A New Perspective on the Wrapping Effect in Interval Methods for Initial Value Problems for Ordinary Differential Equations*, pages 219–263. Springer Vienna, Vienna, 2001.